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TRANSIENT CONDUCTION IN A TWO-DIMENSIONAL COMPOSITE SLAB—I. THEORETICAL DEVELOPMENT OF TEMPERATURE MODES

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Abstract—The response of a two-dimensional, multi-layer composite conducting slab, to a sudden change in the temperature of the surrounding fluid, has been analysed. The solution is of the form of a coupled infinite series in the two space dimensions with an exponential time dependency and in the case of three- and two-layer slabs, expressions are given in a form suitable for desk calculations.

NOMENCLATURE

A_j, B_j, C_j, D_j	constants in the series solution
a_j	thermal diffusivity of the jth layer of the slab
B_t	Biot number, hx_J/k_J
c_j	specific heat of the jth layer of the slab
E_j, F_j, G_j, H_j	defined by equation (21)
h	heat transfer coefficient from the surface of the slab
J	number of layers in the slab
k_j	thermal conductivity of the jth layer of the slab
T	temperature in the slab
t	time
X	defined in equation (24)
x	position across the slab
x_j	position of interface between the jth and $(j+1)$th layers
x_J	thickness of the slab
y	position along the slab
y_1	length of the slab

Greek symbols

α	normalized thermal diffusivity
β	normalized thermal conductivity
γ	normalized slab length
$\Delta_{n,0} = 1, n = 0$; $= 0, n \neq 0$	Dirac delta function
η	normalized position along the slab
θ	normalized time
λ	transverse eigenvalue
μ	longitudinal eigenvalue
ρ_j	density of the jth layer of the slab
τ_j	time constant in the jth layer of the slab
ψ	normalized position across the slab

INTRODUCTION

THE NEED for this work arose from a solar energy space-heating system, in which a rockbed thermal energy store is located beneath and in contact with a concrete slab floor [1–3]. Energy transfer from the rockbed store to the house is by conduction through the floor. It is necessary to determine a rockbed depth which will give adequate storage capacity, and the rate of decay of any temperature variations along the floor after airflow through the rockbed has ceased, since this will also influence both the design of the rockbed and the occupants' acceptance of the system.

For lightweight houses, which are common in a number of countries, the thermal mass of the house is negligible relative to that of the slab and the rockbed. The surface area of the vertical edges of the slab and the rockbed is very much less than the floor area and if the edges are insulated, then the heat transfer through them could be neglected. The heat transfer from the rockbed to the ground must necessarily be very small after the system has been operating for some time and for other than very thin rockbeds, which would have very small energy storage capacities, the heat transfer from the ground to the house will be small because of the thermal resistance of the rockbed.

These observations suggest that a reasonable representation of the temperature decay in this floor-space-heating system, after the forced airflow through the rockbed has ceased, would be given by a two-dimensional composite slab which is subjected to a sudden change in the environmental temperature. The composite slab is insulated at the bottom and the edges so that energy is transferred only through the top surface, as shown in Fig. 1, and it is infinite in the third dimension.

The one-dimensional infinite composite slab has been studied quite extensively [4–11], but there appears to have been no need to consider the two-dimensional case. There are two forms of composite slab which are of interest for the space-heating system mentioned above:

(1) the two layers of the slab in perfect thermal contact;

(2) a resistance between the concrete and the rockbed, which would be the case if the rockbed settled over a period and left a small air gap beneath the

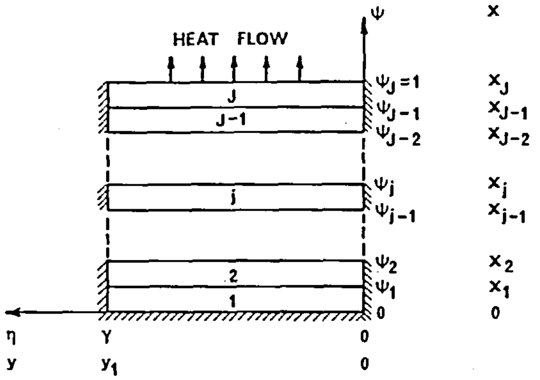

FIG. 1. The two-dimensional multi-layer composite slab.

concrete, or if the concrete were poured onto a plastic membrane laid over the rockbed.

A general solution for any multi-layer two-dimensional slab, whose layers are in perfect thermal contact, is developed here and then particular solutions for the above two cases are given. The heat transfer resistance in case (2) is accommodated by solving a three-layer slab and choosing an appropriately thin layer between the two outer layers.

THE J-LAYER COMPOSITE SLAB

The space coordinates and their values at the boundaries are shown in Fig. 1. The temperature T at the point (x, y) in the jth layer at time $t \geqslant 0$, is governed by the two-dimensional diffusion equation, namely

$$\frac{\partial T}{\partial t} = a_j\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right),$$

where a_j is the thermal diffusivity of the jth layer and is given by

$$a_j = \frac{k_j}{\rho_j c_j}, \tag{1}$$

where k, ρ, and c are the thermal conductivity, the density and specific heat, respectively. The temperature T can be measured relative to that of the surroundings for $t \geqslant 0$ so that for clarity in the equations, the temperature of the surroundings may be set to zero for $t \geqslant 0$.

The problem has been non-dimensionalized in the following way: let

$$\psi = \frac{x}{x_J},\quad \eta = \frac{y}{x_J},\quad \theta = \frac{t a_J}{x_J^2},\quad \gamma = \frac{y_1}{x_J},$$

$$\alpha_j = \frac{a_j}{a_J},\quad \beta_j = \frac{k_j}{k_J};$$

then the governing equation can be written as

$$\frac{\partial T}{\partial \theta} = \alpha_j\left(\frac{\partial^2 T}{\partial \psi^2} + \frac{\partial^2 T}{\partial \eta^2}\right). \tag{2}$$

Consider a solution of equation (2) of the form

$$T = \exp(-\alpha_j \tau_j^2 \theta)(A_j \cos \lambda_j \psi + B_j \sin \lambda_j \psi) \times (C_j \cos \mu_j \eta + D_j \sin \mu_j \eta), \tag{3}$$

where the constant C_j is included explicitly for use with separable initial temperature distributions, then equation (2) requires that

$$\tau_j^2 = \lambda_j^2 + \mu_j^2. \tag{4}$$

The boundary conditions are, for $\theta > 0$

$$\eta = 0 \text{ or } \gamma,\quad \frac{\partial T}{\partial \eta} = 0, \tag{5}$$

$$\psi = 0,\quad \frac{\partial T}{\partial \psi} = 0, \tag{6}$$

$$\psi = \psi_j,\quad T_{(\text{layer } j)} = T_{(\text{layer } j+1)}, \tag{7}$$

and

$$\beta_j \frac{\partial T}{\partial \psi} = \beta_{j+1} \frac{\partial T}{\partial \psi}, \tag{8}$$

$$\psi = \psi_J = 1,\quad \beta_J \frac{\partial T}{\partial \psi} = -B_t T, \tag{9}$$

where

$$B_t = \frac{h x_J}{k_J}, \tag{10}$$

and h is the heat transfer coefficient from the surface of the slab to the surroundings.

Boundary condition (5) requires, from equation (3), that $D_j = 0$, and that

$$\mu_j \sin \mu_j \gamma = 0,$$

which gives

$$\mu_j = \mu_n = n\pi/\gamma,\quad n = 0, 1, 2, \ldots \tag{11}$$

Hence there is an infinite set of eigenvalues in the η-coordinate which satisfy equation (2) and the boundary conditions, and they have the same value in all the layers of the composite slab.

Boundary condition (6) requires, from equation (3), that $B_1 = 0$. Boundary condition (7) requires, from equation (3)

$$\exp(-\alpha_j \tau_j^2 \theta)(A_j \cos \lambda_j \psi_j + B_j \sin \lambda_j \psi_j) C_j = \exp(-\alpha_{j+1} \tau_{j+1}^2 \theta)(A_{j+1} \cos \lambda_{j+1} \psi_j + B_{j+1} \sin \lambda_{j+1} \psi_j) C_{j+1}. \tag{12}$$

Since this equation is true for all values of θ

$$\alpha_j \tau_j^2 = \alpha_{j+1} \tau_{j+1}^2,$$

or, substituting from equation (4)

$$\alpha_j(\lambda_j^2 + \mu^2) = \alpha_{j+1}(\lambda_{j+1}^2 + \mu^2). \tag{13}$$

Then equation (12) becomes

$$(A_j \cos \lambda_j \psi_j + B_j \sin \lambda_j \psi_j) C_j = (A_{j+1} \cos \lambda_{j+1} \psi_j + B_{j+1} \sin \lambda_{j+1} \psi_j) C_{j+1}. \tag{14}$$

Similarly, condition (8) requires equation (13) together with

$$\beta_j\lambda_j(-A_j \sin \lambda_j\psi_j + B_j \cos \lambda_j\psi_j)C_j = \beta_{j+1}\lambda_{j+1}(-A_{j+1} \sin \lambda_{j+1}\psi_j + B_{j+1} \cos \lambda_j\psi_j)C_{j+1}. \quad (15)$$

Equations (14) and (15) can be combined into the following matrix equation

$$\begin{bmatrix} 1 & 0 \\ 0 & \beta_j\lambda_j \end{bmatrix} \Omega[\lambda_j\psi_j] \begin{bmatrix} A_j \\ B_j \end{bmatrix} C_j = \begin{bmatrix} 1 & 0 \\ 0 & \beta_{j+1}\lambda_{j+1} \end{bmatrix} \times \Omega[\lambda_{j+1}\psi_j] \begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} C_{j+1}, \quad (16)$$

where

$$\Omega[\lambda_j\psi_j] = \begin{bmatrix} \cos \lambda_j\psi & \sin \lambda_j\psi \\ -\sin \lambda_j\psi & \cos \lambda_j\psi \end{bmatrix}. \quad (17)$$

Equation (16) can be rewritten as

$$C_{j+1}\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix} = \Omega^{-1}[\lambda_{j+1}\psi_j] \begin{bmatrix} 1 & 0 \\ 0 & \beta_{j+1}\lambda_{j+1} \end{bmatrix}^{-1} \times \begin{bmatrix} 1 & 0 \\ 0 & \beta_j\lambda_j \end{bmatrix} \Omega[\lambda_j\psi_j] \begin{bmatrix} A_j \\ B_j \end{bmatrix} C_j, \quad (18)$$

and condition (9) can be expressed in the following matrix form

$$[B_t \quad 1] \begin{bmatrix} 1 & 0 \\ 0 & \beta_J\lambda_J \end{bmatrix} \Omega[\lambda_J\psi_J] \begin{bmatrix} A_J \\ B_J \end{bmatrix} = 0. \quad (19)$$

Now

$$\begin{bmatrix} 1 & 0 \\ 0 & \beta_{j+1}\lambda_{j+1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \beta_j\lambda_j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\beta_j\lambda_j}{\beta_{j+1}\lambda_{j+1}} \end{bmatrix}, \quad (20)$$

and

$$\Omega[\lambda_j\psi_j]\Omega^{-1}[\lambda_j\psi_{j-1}] = \Omega[\lambda_j(\psi_j - \psi_{j-1})]. \quad (21)$$

Then successively substituting equation (18) into equation (19), and dividing through by A_1C_1, gives

$$[B_t \quad 1] \begin{bmatrix} 1 & 0 \\ 0 & \beta_J\lambda_J \end{bmatrix} \Omega[\lambda_J(\psi_J - \psi_{J-1})] \times \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\beta_{J-1}\lambda_{J-1}}{\beta_J\lambda_J} \end{bmatrix} \Omega[\lambda_{J-1}(\psi_{J-1} - \psi_{J-2})] \times \dots \times \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\beta_2\lambda_2}{\beta_3\lambda_3} \end{bmatrix} \Omega[\lambda_2(\psi_2 - \psi_1)] \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\beta_1\lambda_1}{\beta_2\lambda_2} \end{bmatrix} \times \Omega[\lambda_1\psi_1] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0, \quad (22)$$

where, from equation (13)

$$\lambda_j = \left[\frac{1}{\alpha_j}(\lambda_J^2 + \mu^2) - \mu^2\right]^{1/2}. \quad (23)$$

The negative root of equation (23) does not give additional independent solutions and will be ignored.

Hence, the need to match the temperatures and heat flows at the interfaces requires, from equation (22), that for each value μ_n of the infinite set of eigenvalues in the η-coordinate, there is an associated infinite set of eigenvalues in the ψ-coordinate λ_{jnm} where $m = 1, 2, 3, \dots$; from equation (23), the eigenvalues in the ψ-coordinate are different for each of the layers of the slab. It is interesting to note that whereas for a homogeneous two-dimensional slab it is sometimes possible to form a complete solution as the product of two independent one-dimensional solutions [4], equation (22) shows that this is never possible for a composite slab because the eigenvalues λ_{jnm} for the different layers in the ψ-coordinate across the slab are coupled to each other through the associated eigenvalue μ_n in the η-coordinate along the slab through equation (23). The complete description of the temperature in any one of the layers of the slab is

$$T = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \exp\{-(\lambda_{jnm}^2 + \mu_n^2)\theta\} C_{jn} \cos \mu_n\eta \times (A_{jnm} \cos \lambda_{jnm}\psi + B_{jnm} \sin \lambda_{jnm}\psi). \quad (24)$$

Generally, the subscripts n and m will not be written but they are of course always implied.

The coefficients C_jA_j and C_jB_j can be expressed in terms of C_1A_1 as follows:

$$C_j\begin{bmatrix} A_j \\ B_j \end{bmatrix} = \begin{bmatrix} E_j & F_j \\ G_j & H_j \end{bmatrix} \begin{bmatrix} A_1 \\ 0 \end{bmatrix} C_1, \quad (25)$$

where from equation (15)

$$\begin{bmatrix} E_j & F_j \\ G_j & H_j \end{bmatrix} = \Omega^{-1}[\lambda_j\psi_{j-1}] \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\beta_{j-1}\lambda_{j-1}}{\beta_j\lambda_j} \end{bmatrix} \times \Omega[\lambda_{j-1}(\psi_{j-1} - \psi_{j-2})] \times \dots \times \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{\beta_1\lambda_1}{\beta_2\lambda_2} \end{bmatrix} \Omega[\lambda_1\psi_1]. \quad (26)$$

It is to be noted that E_j and G_j are functions of β_i, λ_i and ψ_{i-1}, where $i = 1, \dots, j$, only, and are independent of all the coefficients A_j and B_j, $j = 1, \dots, J$; this property of E_j and G_j is used below to determine A_j and B_j.

If $T_0(\psi, \eta)$ is the temperature distribution in the slab at $\theta = 0$, and noting that from equation (25)

$$\frac{B_j}{A_j} = \frac{G_j}{E_j}, \quad (27)$$

then from equations (24) and (25)

$$T_0 = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{1n}A_{1nm} \cos \mu_n\eta \cdot E_{jnm}X_{nm}, \quad (28)$$

where

$$X_{nm} = \cos \lambda_{jnm}\psi + \frac{G_{jnm}}{E_{jnm}} \sin \lambda_{jnm}\psi. \quad (29)$$

The coefficients $C_{1n}A_{1nm}$ can be found in the following way. Multiply equation (28) by

$$(\beta_j/\alpha_j)E_{jpq}X_{pq}\cos\mu_p\eta,$$

and integrate over the slab to give

$$\int_0^\gamma\int_0^1 T_0\frac{\beta_j}{\alpha_j}E_{jpq}X_{pq}\cos\mu_p\eta\,d\psi\,d\eta = \int_0^\gamma\int_0^1\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\beta_j}{\alpha_j}C_{1n}A_{1nm}\cos\mu_p\eta \times\cos\mu_n\eta\cdot E_{jpq}E_{jnm}X_{pq}X_{nm}\,d\psi\,d\eta. \quad (30)$$

Interchange the order of summation and integration, and note that

$$\int_0^\gamma\cos\mu_p\eta\cos\mu_n\eta\,d\eta = 0 \quad \text{if} \quad n\neq p,$$

then equation (30) becomes

$$\int_0^\gamma\int_0^1 T_0\frac{\beta_j}{\alpha_j}E_{jnq}X_{nq}\cos\mu_n\,d\psi\,d\eta = \int_0^\gamma\cos^2\mu_n\eta\,d\eta\sum_{m=1}^{\infty}\left\{C_{1n}A_{1nm}\times\sum_{j=1}^{J}\left(\frac{\beta_j}{\alpha_j}E_{jnq}E_{jnm}\int_{\psi_{j-1}}^{\psi_j}X_{nq}X_{nm}\,d\psi\right)\right\}. \quad (31)$$

Now from equation (13)

$$\alpha_j(\lambda_{jnm}^2-\lambda_{jnq}^2) = \lambda_{Jnm}^2-\lambda_{Jnq}^2, \quad (32)$$

and then it can be shown that [4]

$$(\lambda_{Jnm}^2-\lambda_{Jnq}^2)\frac{\beta_j}{\alpha_j}E_{jnq}E_{jnm}\int_{\psi_{j-1}}^{\psi_j}X_{nq}X_{nm}\,d\psi = \beta_jE_{jnq}E_{jnm}\left[X_{nm}\frac{dX_{nq}}{d\psi}-X_{nq}\frac{dX_{nm}}{d\psi}\right]_{\psi_{j-1}}^{\psi_j} \quad (33)$$

From condition (7)

$$E_{jnm}X_{nm}(\lambda_j\psi_j) = E_{j+1,nm}X_{nm}(\lambda_{j+1}\psi_j), \quad (34)$$

and from condition (8)

$$\beta_jE_{jnm}\frac{dX_{nm}}{d\psi}\bigg|_{\lambda_j,\psi_j} = \beta_{j+1}E_{j+1,nm}\frac{dX_{nm}}{d\psi}\bigg|_{\lambda_{j+1},\psi_j}. \quad (35)$$

Also, from condition (6)

$$\frac{dX_{nm}}{d\psi}\bigg|_{\psi=0} = 0, \quad (36)$$

and from condition (9)

$$\frac{dX_{nm}}{d\psi}\bigg|_{\psi=1} = -B_tX_{nm}\bigg|_{\psi=1} \quad (37)$$

Then, by virtue of equations (33)–(37)

$$(\lambda_{Jnm}^2-\lambda_{Jnq}^2)\sum_{j=1}^{J}\times\left(\frac{\beta_j}{\alpha_j}E_{jnq}E_{jnm}\int_{\psi_{j-1}}^{\psi_j}X_{nq}X_{nm}\,d\psi\right) = 0, \quad (38)$$

and hence

$$\int_0^1\frac{\beta_j}{\alpha_j}E_{jnq}E_{jnm}X_{pq}X_{nm}\,d\psi = 0 \quad \text{if} \quad m\neq q. \quad (39)$$

Noting that

$$\int_0^\gamma\cos^2\mu_n\eta\,d\eta = \frac{\gamma}{2-\Delta_{n,0}} \quad (40)$$

where

$$\Delta_{n,0} = 1 \quad \text{if} \quad n = 0,$$
$$= 0 \quad \text{if} \quad n\neq 0,$$

then substituting equations (39) and (40) into equation (28) gives

$$C_{1n}A_{1nm} = \frac{\int_0^\gamma\int_0^1 T_0\cos\mu_n\eta\cdot(\beta_j/\alpha_j)E_{jnm}X_{nm}\,d\psi\,d\eta}{[\gamma/(2-\Delta_{n,0})]\int_0^1(\beta_j/\alpha_j)E_{jnm}^2X_{nm}^2\,d\psi}. \quad (41)$$

The integral in the denominator of equation (41) can be evaluated as

$$\int_0^1\frac{\beta_j}{\alpha_j}E_j^2X^2\,d\psi = \sum_{j=1}^{J}\frac{\beta_j}{\alpha_j}\times\left\{\begin{array}{l}\frac{1}{2}(\psi_j-\psi_{j-1})(E_j^2+G_j^2)+\frac{1}{2}\lambda_j\cos[\lambda_j(\psi_j\\ +\psi_{j-1})]\sin[\lambda_j(\psi_j-\psi_{j-1})](E_j^2-G_j^2)\\ +\frac{1}{\lambda_j}\sin[\lambda_j(\psi_j+\psi_{j-1})]\sin[\lambda_j(\psi_j\\ -\psi_{j-1})]E_jG_j\end{array}\right\}. \quad (42)$$

The transient temperature response of the slab is fully described by equations (22), (41) and (18) for all initial temperature distributions within the slab. If this happens to be separable in the two space dimensions, then equation (41) can be separated and C_{jn} and A_{jnm} can be determined independently.

The use of an initial temperature distribution which is the product of two one-dimensional distributions, provides the following useful check on any implementation of the solution. At time $\theta = 0$, and the point $\psi = \eta = 0$ in the slab, from equation (24)

$$T_c(0,0,0) = \sum_{n=0}^{\infty}\left\{C_{1n}\sum_{m=1}^{\infty}A_{1nm}\right\}, \quad (43)$$

and hence in this case, $\sum_{m=1}^{\infty}A_{1nm}$ must be a constant for all values of 'n'.

THE THREE-LAYER COMPOSITE SLAB

In this case, $\beta_3 = \alpha_3 = 1$ and $J = 3$ and the transverse eigenvalues λ_{jnm} can be found from the following reduced form of equation (22)

$$[B_t \quad 1]\begin{bmatrix}1 & 0\\ 0 & \beta_3\lambda_3\end{bmatrix}\Omega[\lambda_3(\psi_3-\psi_2)]\begin{bmatrix}1 & 0\\ 0 & \dfrac{\beta_2\lambda_2}{\beta_3\lambda_3}\end{bmatrix}$$

$$\times\,\Omega[\lambda_2(\psi_2-\psi_1)]\begin{bmatrix}1 & 0\\ 0 & \dfrac{\beta_1\lambda_1}{\beta_2\lambda_2}\end{bmatrix}\Omega[\lambda_1\psi_1]\begin{bmatrix}1\\0\end{bmatrix}=0,$$

which can be expressed algebraically as

$$[B_t-\lambda_3\tan\lambda_3(1-\psi_2)]\times\left[1-\frac{\beta_1\lambda_1}{\beta_2\lambda_2}\tan\lambda_2(\psi_2-\psi_1)\tan\lambda_1\psi_1\right]-\left[1+\frac{B_t}{\lambda_3}\tan\lambda_3(1-\psi_2)\right][\beta_2\lambda_2\times\tan\lambda_2(\psi_2-\psi_1)+\beta_1\lambda_1\tan\lambda_1\psi_1]=0. \quad (44)$$

From equation (25), $E_1 = 1$ and $G_1 = 0$

$$E_2=\cos\lambda_2\psi_1\cos\lambda_1\psi_1+\frac{\beta_1\lambda_1}{\beta_2\lambda_2}\sin\lambda_2\psi_1\sin\lambda_1\psi_1. \quad (45)$$

and

$$G_2=\sin\lambda_2\psi_1\cos\lambda_1\psi_1-\frac{\beta_1\lambda_1}{\beta_2\lambda_2}\times\cos\lambda_2\psi_1\sin\lambda_1\psi_1. \quad (46)$$

Also

$$E_3 = R_1\cos\lambda_3\psi_2+R_2\sin\lambda_3\psi_2, \quad (47)$$

and

$$G_3 = R_1\sin\lambda_3\psi_2-R_2\cos\lambda_3\psi_2, \quad (48)$$

where

$$R_1=\cos\lambda_2(\psi_2-\psi_1)\cos\lambda_1\psi_1-\frac{\beta_1\lambda_1}{\beta_2\lambda_2}\times\sin\lambda_2(\psi_2-\psi_1)\sin\lambda_1\psi_1, \quad (49)$$

and

$$R_2=\frac{\beta_2\lambda_2}{\beta_3\lambda_3}\left[\sin\lambda_2(\psi_2-\psi_1)\cos\lambda_1\psi_1+\frac{\beta_1\lambda_1}{\beta_2\lambda_2}\times\cos\lambda_2(\psi_2-\psi_1)\sin\lambda_1\psi_1\right]. \quad (50)$$

$C_{1n}A_{1nm}$ can then be found using equations (41) and (42) with $J = 3$.

In this form, the solution can be evaluated on a programmable pocket calculator such as a Hewlett Packard HP67. The determination of the transverse eigenvalues from equation (44) is easily accommodated on the calculator, as is the calculation of E_2, G_2, E_3 and G_3, but the calculation of T using equation (24) can be tedious if many values of θ require the use of many terms of the series. However, as θ increases, only the first two or three terms of the series become significant and the programmable pocket calculator is a practical tool for use in this solution.

THE TWO-LAYER COMPOSITE SLAB

In this case, $\beta_2 = \alpha_2 = 1$ and equation (22) can be expressed as

$$[B_t \quad 1]\begin{bmatrix}1 & 0\\ 0 & \beta_2\lambda_2\end{bmatrix}\Omega[\lambda_2(\psi_2-\psi_1)]\times\begin{bmatrix}1 & 0\\ 1 & \dfrac{\beta_1\lambda_1}{\beta_2\lambda_2}\end{bmatrix}\Omega[\lambda_1\psi_1]\begin{bmatrix}1\\0\end{bmatrix}=0,$$

and in algebraic form as

$$B_t-B_t\frac{\beta_1\lambda_1}{\lambda_2}\tan\lambda_2(\psi_2-\psi_1)\tan\lambda_1\psi_1-\lambda_2\tan\lambda_2(\psi_2-\psi_1)-\beta_1\lambda_1\tan\lambda_1\psi_1=0. \quad (51)$$

E_2 and G_2 are given by equations (45) and (46) and $C_{1n}A_{1nm}$ can be found using equations (41) and (42) with $J = 2$.

THE HOMOGENEOUS SLAB

In this case, $\beta_1 = \alpha_1 = 1$ and equation (22) gives

$$[B_t \quad 1]\begin{bmatrix}1 & 0\\ 0 & \beta_1\lambda_1\end{bmatrix}\Omega[\lambda_1\psi_1]\begin{bmatrix}1\\0\end{bmatrix}=0,$$

which reduces to the following well-known transcendental equation for λ_1, namely

$$\lambda_1\tan\lambda_1-B_t=0. \quad (52)$$

In this case, as is well known, the eigenvalues λ_1 associated with the transverse coordinate ψ are independent of the eigenvalues μ associated with the longitudinal coordinate η. Since B_1 is zero, equation (24) can be written as

$$T_1=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\exp\{-(\lambda_{1m}^2+\mu_n^2)\theta\}C_{1n}\times\cos\mu_n\eta\cdot A_{1m}\cos\lambda_{1m}\psi, \quad (53)$$

and the reduced form of equation (41) is

$$C_{1n}A_{1m}=\frac{\displaystyle\int_0^{\gamma}\int_0^1 T_0\cos\lambda_{1m}\psi\cdot\cos\mu_n\eta\,d\psi\,d\eta}{[\gamma/(2-\Delta_{n,0})]\frac{1}{2}\{1+[(\sin\lambda_{1m}\cos\lambda_{1m})/\lambda_{1m}]\}}. \quad (54)$$

If the initial temperature distribution in the slab is separable, with $T_0 = T_1(\psi)T_2(\eta)$, then C_{1n} is given by

$$C_{1n}=\frac{2-\Delta_{n,0}}{\gamma}\int_0^{\gamma}T_2\cos\mu_n\eta\,d\eta, \quad (55)$$

and

$$A_{1m}=\frac{\displaystyle\int_0^1 T_1\cos\lambda_{1m}\psi\,d\psi}{\frac{1}{2}\{1+[(\sin\lambda_{1m}\cos\lambda_{1m})/\lambda_{1m}]\}},$$

which is the familiar form found in textbooks [12].

Hence, as was stated earlier, if the initial temperature distribution is the product of two one-dimensional distributions in the two space coordinates then, for the homogeneous slab, the complete solution (53) is the product of the two one-dimensional solutions.

CONCLUSIONS

A complete description has been derived for the change in temperature with time at any point within a two-dimensional, multi-layer composite slab, which is coupled to its environment through one surface of one layer only, after the temperature of the environment has suffered a step change. The solution has the form of the product of an exponential time dependency and two spatial eigenfunctions. The eigenfunctions associated with the space coordinate along the layers of the slab are independent of the layers, whereas the eigenfunctions associated with the space coordinate across the slab are different in each layer. For each longitudinal eigenvalue there is an infinite set of transverse eigenvalues so that the transient response of the two-dimensional composite slab can never be expressed as the product of two one-dimensional solutions, whereas this is sometimes possible for a homogeneous slab. A physical interpretation of some aspects of the solution is given elsewhere [13].

The transcendental equation which determines the eigenvalues is expressed in matrix form for the general multi-layer slab but algebraic forms are given for the three- and two-layer slabs; in this form, the solutions are suitable for desk calculations with a programmable pocket calculator. It is shown that the derived solution for the multi-layer composite slab reduces to the well-known form for the homogeneous slab.

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CONDUCTION VARIABLE DANS UNE PLAQUE COMPOSITE BIDIMENSIONNELLE. DEVELOPPEMENT THEORIQUE DES MODES DE TEMPERATURE

Résumé—On analyse la réponse d'une plaque bidimensionnelle, composite à plusieurs couches, à un changement brusque de température du fluide environnant. La solution est sous la forme de séries infinies couplées dans les deux dimensions spatiales avec une dépendance exponentielle du temps et, dans le cas de trois et deux couches, des expressions sont donées dans une forme utilisable pour les calculs de bureau.

INSTATIONÄRE WÄRMELEITUNG IN EINER ZWEIDIMENSIONALEN GESCHICHTETEN PLATTE—I. THEORETISCHE ABLEITUNG DER LÖSUNGEN FÜR DIE TEMPERATUR

Zusammenfassung—Die Sprungantwort einer zweidimensionalen, mehrfach geschichteten wärmeleitenden Platte auf eine sprunghafte Änderung der Temperatur des umgebenden Fluids wurde untersucht. Die Lösung hat die Form von gekoppelten unendlichen Reihen in den zwei Raumdimensionen mit einer exponentiellen Zeitabhängigkeit, im Fall von drei- und zweischichtigen Platten werden Lösungen von einer Form angegeben, die für Berechnungen mit dem Tischrechner geeignet ist.

НЕУСТАНОВИВШАЯСЯ ПЕРЕДАЧА ТЕПЛА ТЕПЛОПРОВОДНОСТЬЮ В ДВУМЕРНОЙ КОМПОЗИТНОЙ ПЛИТЕ—I. ТЕОРЕТИЧЕСКОЕ ОПИСАНИЕ ТЕМПЕРАТУРНЫХ МОД

Аннотация—Проведен анализ влияния мгновенного изменения температуры окружающей среды на передачу тепла теплопроводностью в двумерной композитной плите. Решение представлено в виде взаимосвязанных бесконечных рядов по двум пространственным координатам с экспоненциальной временной зависимостью. Для случая трех- или двухслойных плит выражения даны в удобном для численных расчетов виде.